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Surfaces of Rotation with Constant Measure of Curvature and their Representation on the Hyperbolic (Cayley's) Plane.

By Geo. F. Metzler, *Odessa, Ont.*

In Crelle, vols. 19 and 20, Minding has shown that it is easy to obtain the formulas which express the relations between the sides and angles of a triangle of which the sides are geodesic lines on a surface of rotation with constant measure of curvature.

It is only necessary to replace the radius a of the sphere in the formulas of the ordinary spherical trigonometry by $a\sqrt{-1}$.

That this is also true for the formula expressing the area of the triangle is not stated, and as that is the formula which concerns us here, a proof of the same will be given by means of polar and rectangular coordinates. I do not know that such proof has yet been published.

It will be assumed that such definitions and fundamental equations as are to be found in Salmon's "Geometry of Three Dimensions," Joachimsthal's "Anwendung der Differentiale Rechnung" and similar works, are well known.

If the equations of the surface are

$$u = f(r) - z = 0, \quad r^2 = x^2 + y^2, \tag{1}$$

and we designate the derivates of u as follows:

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}, \quad -1 = \frac{\partial u}{\partial z}, \quad v = \frac{\partial^2 u}{\partial x^2},$$

$$s = \frac{\partial^2 u}{\partial x \partial y}, \quad t = \frac{\partial^2 u}{\partial y^2} \text{ and } f' = \frac{\partial z}{\partial r} = \frac{\partial u}{\partial r},$$

then R_1 and R_2, the principal radii of curvature, are the roots of the equation

$$R^2(vt-s^2)-R\left[(1+q^2)v-2pqs+(1+p^2)t\right]\sqrt{1+p^2+q^2} + (1+p^2+q^2)^2=0. \tag{2}$$

Thus the measure of curvature $\frac{1}{R_1 R_2}$ is expressed by

$$\frac{1}{R_1R_2}=\frac{vt-s^2}{(1+p^2+q^2)^2}. \tag{3}$$

From (1) we obtain

$$p=\frac{x}{r}f',\quad q=\frac{y}{r}f',\quad v=\frac{x^2}{r^2}f''+\frac{y^2}{r^3}f',$$

$$s=\frac{xy}{r^2}f''-\frac{xy}{r^3}f'\quad,\quad t=\frac{y^2}{r^2}f''+\frac{x^2}{r^3}f'.$$

Then

$$vt-s^2=\left(\frac{x^2}{r^2}f''+\frac{y^2}{r^3}f'\right)\left(\frac{y^2}{r^2}f''+\frac{x^2}{r^3}f'\right)-\left(\frac{xy}{r^2}f''-\frac{xy}{r^3}f'\right)^2=\frac{f'f''}{r},$$

$$(1+p^2+q^2)^2=(1+f'^2)^2.$$

Then follows

$$\frac{1}{R_1R_2}=\frac{f'f''}{r(1+f'^2)^2}=-\frac{1}{2r}\frac{d}{dr}\left(\frac{1}{1+f'^2}\right). \tag{4}$$

If the surface has a constant measure of curvature, $\pm\frac{1}{a^2}$ say, then it follows from (4) that

$$-\frac{1}{1+f'^2}=\pm\int\frac{2r\,dr}{a^2}=\pm\frac{r^2}{a^2}-b^2,$$

where $-b^2$ is the constant of integration. Then

$$1+f'^2=\frac{a^2}{a^2b^2\mp r^2},$$

$$f'^2=\frac{a^2(1-b^2)\pm r^2}{a^2b^2\mp r^2},$$

and

$$dz=dr\sqrt{\frac{a^2(1-b^2)\pm r^2}{a^2b^2\mp r^2}}. \tag{5}$$

Equation (5) gives the meridian curves of six different forms of surfaces of rotation. According to the values given to b^2 we obtain one of three different forms by using the upper signs, and also one of three other different forms when the lower signs are used.

Conversely, it is easy to show that all surfaces of rotation whose meridian curves are given by (5) are surfaces of constant curvature.

Sections of the surfaces through the axis of rotation and a pair of meridian curves would appear as in the following figures.*

Surfaces with Positive Measure of Curvature.

Fig 1, b<1

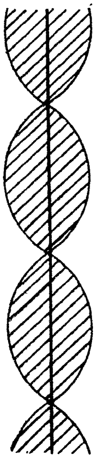


Fig 2, b=1

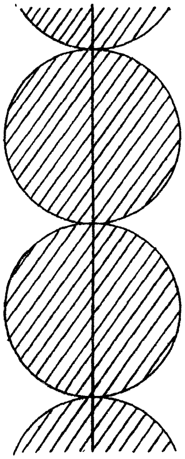
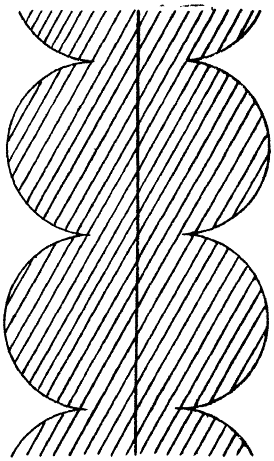


Fig. 3, b>1



From the equation

$$\frac{dz}{dr}=\sqrt{\frac{a^2(1-b^2)+r^2}{a^2b^2-r^2}}$$

we see that the greatest real value for r^2 is a^2b^2, for which $dr=0$. dz vanishes for $r^2=a^2b^2-a^2$. In Fig. 1, however, this value of r is imaginary, and 0 is the minimum real value of r.

Thus the surfaces consist of an infinite number of parts, each part having the axis of z as axis of rotation. The area of one part of these surfaces is given by

$$\left.\begin{aligned} &2\int_{r^2=a^2b^2-a^2}^{r^2=a^2b^2} 2\pi r ds, \quad ds=\sqrt{1+\left(\frac{dz}{dr}\right)^2}dr, \\ &2\int 2\pi r ds = 4\pi\int_{a^2b^2-a^2}^{a^2b^2}\frac{ardr}{\sqrt{a^2b^2-r^2}} = -\left[4\pi a\sqrt{a^2b^2-r^2}\right]_{a^2b^2-a^2}^{a^2b^2} \\ &\qquad = 4\pi a^2. \end{aligned}\right\} \qquad (6)$$

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* Klein, Nicht-Euklidische Geometrie, p. 186.

If in Fig. 1 we do not consider the imaginary part of the surface, 0 becomes the inferior limit and the area is

$$4\pi a^2 b,$$

and this is equal to the superficial area of a cylinder of altitude $2a$ circumscribing the surface. It is easy to see that the meridian curves approach in form for Fig. 1 those on a prolate spheroid, for Fig. 3 those on an oblate spheroid, while for Fig. 2 they are circular.

Surfaces with Constant Negative Measure of Curvature.

Fig. 4, $b^2 > 0$

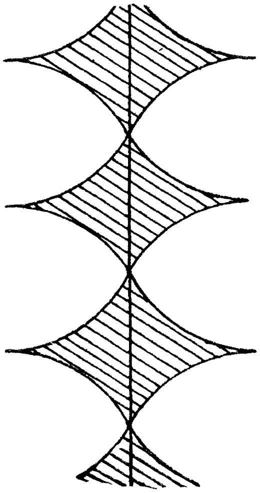


Fig. 5, $b^2 = 0$

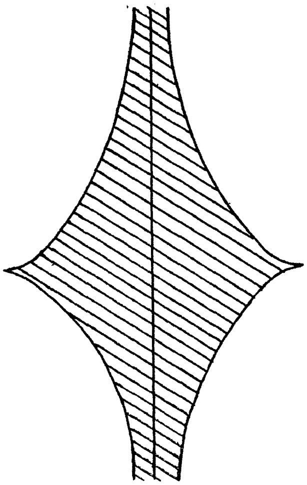
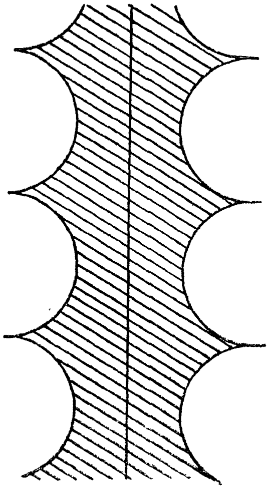


Fig. 6, $b^2 < 0$



The equation for these forms is

$$\frac{dz}{dr} = \sqrt{\frac{a^2(1 - b^2) - r^2}{a^2 b^2 + r^2}},$$

which is the same as that for the other forms with $-a^2$ written for a^2.

In Figs. 4 and 6 we have also a series of similar parts, while in Fig. 5 there is only one part extending to infinity in both directions. It is easy to show that the area of the surface of each part is the same for all cases when we take for limits values of r for which dz and dr vanish. It is

$$4\pi a^2.$$

Otherwise, if we take in Fig. 4 zero as the inferior limit of integration, the area is

$$4\pi a^2(1-b).$$

Thus the area of the real part of Fig. 4 equals the area of the imaginary part of Fig. 1, and *vice versa.*

Now let us seek the formula for the area of any triangle whose sides are geodesics on these surfaces.

When we consider the geodesics projected on a plane perpendicular to the axis of rotation, the equation of the projection in polar coordinates r, ϕ is (compare Joachimsthal, "Anwendung der Diff. Rech.," p. 172)

$$\phi = \nu\int\frac{dr}{r}\sqrt{\frac{1+f'^2}{r^2-\nu^2}}, \tag{7}$$

where

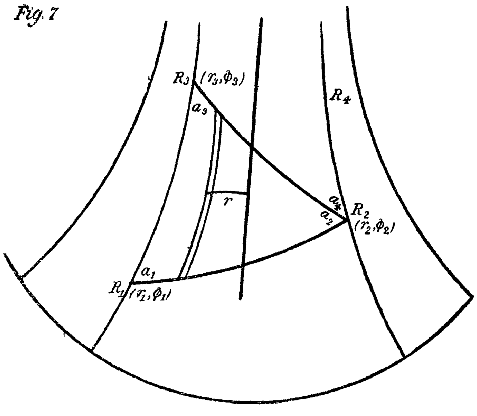
$$1+f'^2 = 1+\left(\frac{dz}{dr}\right)^2 = \frac{a^2}{a^2b^2\mp r^2}.$$

Then

$$d\phi = \frac{a\nu\, dr}{r}\sqrt{\frac{1}{(a^2b^2\mp r^2)(r^2-\nu^2)}}. \tag{8}$$

Through each angle of any triangle a meridian passes. If no side of the triangle be a meridian, the triangle may be divided into two, of which one side is the dividing meridian. Thus we need consider only triangles one of whose sides is a meridian. Let $R_1\,R_2\,R_3$ be such a triangle.

Fig. 7



Let also the coordinates of R_1, R_2 and R_3 be designated by $(r_1\,\phi_1)$, $(r_2\,\phi_2)$, and $(r_3\phi_3)$ and r_{12} designate the values of r along the side $R_1 R_2$, r_{23} its values along $R_2 R_3$. The area Δ of the triangle $R_1 R_2 R_3$ is given by

$$\Delta = \int\int r d\phi\, ds = \int_{\phi_1}^{\phi_2}\int_{r_{23}}^{r_{12}} r d\phi \frac{a dr}{\sqrt{a^2b^2 \mp r^2}}$$
$$= \mp a\int_{\phi_1}^{\phi_2} \sqrt{a^2b^2 \mp r_{12}^2}\, d\phi \pm a\int_{\phi_1}^{\phi_2} \sqrt{a^2b^2 \mp r_{23}^2}\, d\phi.$$

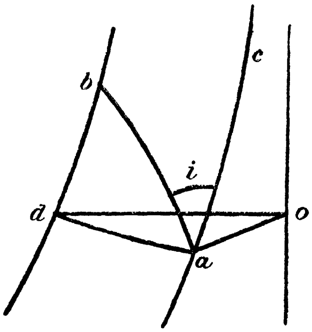
Substituting for $d\phi$ from (8),

$$\Delta = \mp a^2\int_{r_1}^{r_2} \frac{\nu_{12}\, dr_{12}}{r_{12}\sqrt{r_{12}^2 - \nu_{12}^2}} \pm a^2\int_{r_3}^{r_2} - \frac{d\left(\frac{\nu_{23}}{r_{23}}\right)}{\sqrt{1 - \frac{\nu_{23}^2}{r_{23}^2}}}$$
$$= \pm a^2\left[\sin^{-1}\frac{\nu_{12}}{r_2} - \sin^{-1}\frac{\nu_{12}}{r_1} - \sin^{-1}\frac{\nu_{23}}{r_2} + \sin^{-1}\frac{\nu_{23}}{r_3}\right]. \qquad (9)$$

If we take on such a surface two meridian curves ac and db infinitely near together,

$$od = oa = r,$$
$$ab = ds,$$

FIG. 8.



angle $aod = d\phi$, $ad = rd\phi = ab \cos bad$

$$= ab \sin cab$$
$$= ds \sin i,$$

$\nu ds = r^2 d\phi$ (see Joachimsthal ....),

where ν is so chosen that it is positive.

When
$$ds = r d\phi\,, \quad \nu = r\,,$$
then $i = 90°$. Thus ν is the minimum value of r, i. e. its value where the geodesic crosses the meridian perpendicularly.

$$\nu ds = rab \sin i = rds \sin i\,.$$

Therefore
$$\sin^{-1}\frac{\nu}{r} = i\,. \tag{10}$$

Introducing this into (3), we have the interesting result, a_1, a_2, a_3, being the angles of the triangle,

$$\begin{aligned}\Delta &= \pm\, a^2\left[\alpha_2 + \alpha_4 - (\pi - \alpha_1) - \alpha_4 + \alpha_3\right],\\ \Delta &= \pm\, a^2\left[\alpha_1 + \alpha_2 + \alpha_3 - \pi\right].\end{aligned} \tag{11}$$

Thus for all these surfaces of positive measure of curvature the area of a triangle is given by the same formula as holds for the sphere, and when the curvature is negative the proper formula is obtained by replacing a^2 by $-a^2$. (This result is the same as has been found by Beltrami by means of curvilinear coordinates, Beltrami's "Saggio.")

Before considering the conformal representation of these lines, areas, and surfaces, let us study more closely the situation of the geodesics on the surfaces. The case in which the meridian curve is the tractrix (Fig. 5) is simplest and also the most interesting.

Equation (8), the equation of projection, becomes for the traction

$$d\phi = \frac{a\nu\, dr}{r^2\sqrt{r^2 - \nu^2}}\,,$$

then
$$\phi = \frac{1}{\nu}\sqrt{a^2 - \nu^2} - \frac{a}{r\nu}\sqrt{r^2 - \nu^2} + C.$$

Let λ be the angle that the projection of the geodesic makes with the radius r. Then for any of the surfaces with negative measure of curvature,

$$\operatorname{cotan}\lambda = \frac{dr}{rd\phi} = \frac{\sqrt{(a^2b^2 + r^2)(r^2 - \nu^2)}}{a\nu}\,.$$

When $r^2 = a^2(1 - b^2)$ we have an edge of the surface (cuspidal edge). Let λ_c be the value of λ at this edge, then

$$\sin\lambda_c = \frac{\nu}{a\sqrt{1 - b^2}}\,. \tag{12}$$

From every point of this edge geodesics proceed in all directions, each having a definite value of λ_c and consequently a definite value for ν. From the equation for cotan λ we see that ν is the minimum value of r, for which value $\lambda = \frac{\pi}{2}$.

For every value of r from ν to $a\sqrt{1-b^2}$ we obtain a definite ϕ, and therefore a definite x and y, and then from the equation of the surface the corresponding z, so that it is not difficult to form the geodesics on the surface.

Let us now consider the schaar of geodesics which reach their highest points on a certain meridian OA. From this meridian the geodesics considered proceed symmetrically in both directions. Let $\phi = 0$ for this meridian. On it $r = \nu$, $\lambda = \frac{\pi}{2}$ and dr must be positive for each curve. For the surface (Fig. 5) $b^2 = 0$ and

$$\phi = \pm \frac{a}{\nu r}\sqrt{r^2 - \nu^2} = \frac{a^2}{r^2} \text{ cotan } \lambda. \qquad (13)$$

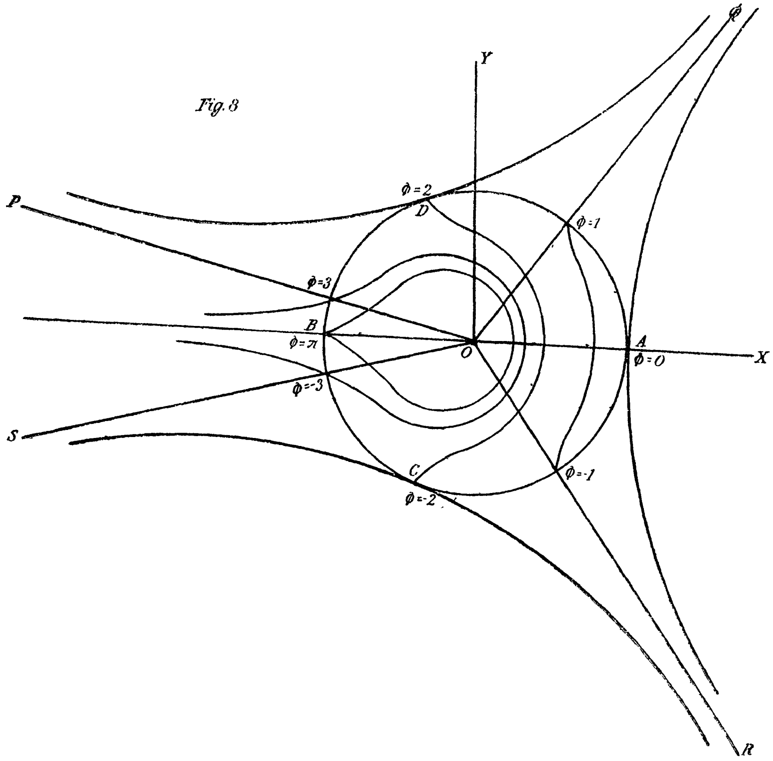


Fig. 8

When ν is very small, λ_c is very small, and for finite r, $d\phi$ is very small, and the geodesic keeps near the meridian; but as r approaches zero, $d\phi$

increases rapidly, so that the curve winds infinitely often around the surface before reaching its highest point.

Let us suppose the upper half of the surface projected on the xy plane passing through the cuspidal edge. This edge projects into the circle $ACBD$, the axis of rotation into the point O, centre of the circle, and the meridian OA into the radius $OA = a$.

From (13) we see that r is infinite when $\pm\phi = \frac{a}{\nu}$. When $\nu = a$ the geodesic touches the cuspidal edge at A and proceeds to infinity, having the lines for which $\phi = \pm 1$ as asymptotes. This is a point to be specially noted in order that correct figures be drawn. It may be stated thus:

Two geodesics which touch the cuspidal edge and meet at infinity have points of contact subtending an angle $\frac{360^\circ}{\pi}$ or 2 at the centre O. Thus the curves touching where $\phi = 0$ and $\phi = 2$ would have the line for which $\phi = 1$ as asymptote. Let $r = a$, then $\nu = a \sin\lambda_c$ and $\phi = \operatorname{cotan}\lambda_c$, then for

$$\left.\begin{array}{llll}
\nu = a & , \quad \phi = 0 & , \quad \lambda_c = 90^\circ & , \\
\nu = \frac{a}{\nu^2} & , \quad \phi = 1 & , \quad \lambda_c = 45^\circ & , \\
\nu = \frac{a}{2} & , \quad \phi = \sqrt{3}, & \quad \lambda_c = 30^\circ & , \\
\nu = a \sin 17^\circ \tfrac{2}{3}, & \quad \phi = \pi & , \quad \lambda_c = 17^\circ \tfrac{2}{3}. & \text{(This last is approximate.)}
\end{array}\right\} \qquad (15)$$

The condition for a point of inflexion is (compare Williamson's "Diff. Cal.," p. 278)

$$rf'' - 2f'^3 - r^2 = 0,$$

when

$$r = \frac{a\nu}{(a^2 - \nu^2\phi^2)^{\frac{1}{2}}} = f(\phi), \quad f' = \frac{a\nu^3\phi}{(a^2 - \nu^2\phi^2)^{\frac{3}{2}}},$$

$$f'' = \frac{a^3\nu^5 + 2a\nu^5\phi^2}{(a^2 - \nu^2\phi^2)^{\frac{5}{2}}}.$$

Therefore

$$a^2\nu^2 = (a^2 - \nu^2\phi^2)^2,$$

$$= \frac{a^4\nu^4}{r^4}$$

and

$$r^2 = a\nu = a^2 \sin\lambda.$$

Thus these geodesics have in general a point of inflection for $r = \sqrt{a\nu} = a\sqrt{\sin\lambda}$.

They are touched at this point by the curves whose tangents are the inflexional tangents of the surface. Let r_t, ϕ be the coördinates of the projection of these curves, while r_g, ϕ are the coordinates of projection of the geodesics. From sections 377, 380 and 381, Salmon's "Geometry of Three Dimensions," the equation of these curves, viz.

$$d\phi = \frac{a\,dr_t}{r_t\sqrt{a^2 - r_t^2}}$$

can be derived.

For the same values of r, ϕ we have for the geodesics

$$d\phi = \frac{\nu a\,dr_g}{r_g^2\sqrt{r_g^2 - \nu^2}}\,.$$

From these equations it follows that

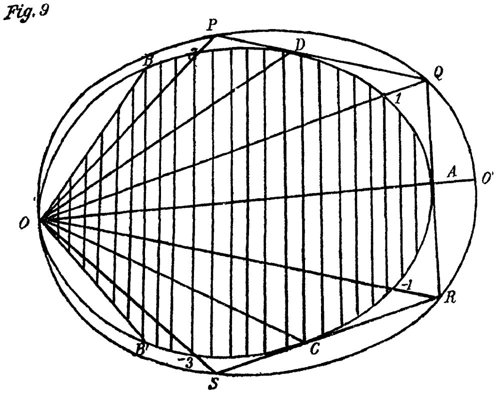
$$\frac{dr_g}{dr_t} = \frac{r\sqrt{r^2 - \nu^2}}{\nu\sqrt{a^2 - r^2}}\,.$$

This fraction takes all values from 0 to ∞ as r increases from ν to a, and equals unity when $r^2 = a\nu$, that is, the two curves touch for this value of r. (This result has been obtained by Mr. Bacharack from another point of view.)

Let us consider the conformal representation of the upper half of this surface on Cayley's plane of hyperbolic measurement.

Let the surface be cut along the meridian which is projected in OB (Fig. 8). The two edges thus formed will be represented by OB and OB'. Since the meridian curves all meet the axis of rotation in the point O for which $z = \infty$, its

Fig. 9



representative O in Fig. 9 should lie on the boundary. (Compare "Nicht-

Euklidische Geometrie," Klein. In fact these celebrated lectures form the source of my knowledge on these questions prior to this investigation.)

Let $OPQRS$ be the boundary of the plane, and $BDACB'$ represent the cuspidal edge. Angles remain unchanged, and geodesics, including meridians, are represented by straight lines. P, Q, R and S represent points infinitely distant where, in Fig. 8, tangent curves at D, A and C touch OP, OQ, OR and OS. Thus the hatched part of the figure is the representation of the upper half of the surface of rotation of the tractrix.

As the area is $2\pi a^2$, when a is small the representation should occupy but a small part of the plane. By moving all the lettered points proportionally towards O', the figure can be made small, care being taken that the curve $BDACB'$ touches the lines PQ, QR, RS at D, A and C respectively. With this figure there is no difficulty in calculating areas.

The formula for a triangle with angles α, β, γ is ("Nicht-Euklidische Geometrie," p. 121)

$$\text{Area} = 4x^2(\pi - \alpha - \beta - \gamma).$$

From Fig. 8 or equations (15) we see that each geodesic subtending an angle 2 at the centre O makes angles equal to $\frac{\pi}{4}$ with radii or meridians through the points where it cuts the cuspidal edge. Then the angles of the triangle OAD are respectively O, $\frac{\pi}{4}$, $\frac{\pi}{4}$ and the area of this triangle is $4x^2\left(\pi - O - \frac{\pi}{2}\right)$ or

$$2\pi x^2.$$

The area of the corresponding triangle on the surface is $\frac{\pi}{2}a^2$, and therefore corresponding areas are to each other as $4x^2$ to a^2.

The whole area is

$$2\pi(4x^2) = 8\pi x^2.$$

The area of the sector $OA1D$ is $8x^2$, which is greater than the triangle OAD, and the area of $OAQD$ is

$$4x^2\left(2\pi - 2\frac{\pi}{2}\right) = 4x^2\pi,$$

and this is greater than the sector, and so for other parts of the figure consistent results are obtained. Such results are impossible when a figure such as $OA1D$ is taken to represent the surface. It is in fact $\frac{1}{\pi}$th of the surface.